

Optimality Conditions for Constrained Optimization: A Systematic Compilation with a Pedagogical Perspective

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Abstract: A systematic compilation of fundamental results for several types of constrained optimization problems—including those with only equality constraints, only inequality constraints, and a mixture of both—is presented, covering the theorems of alternative, Karush-Kuhn-Tucker (KKT) conditions, and strong duality in as complete a form as possible. This exposition is designed to serve as a handy reference for students, offering a self-contained collection of the essential theoretical tools, while also providing instructors with a convenient source for lecture preparation.

Keywords: Constrained Optimization Problems; KKT Conditions; Strong Duality.

1. Introduction

Optimization problems are ubiquitous in engineering practice and scientific research, and they form an essential mathematical foundation for core artificial intelligence courses such as machine learning. Existing textbooks on optimization often lack completeness and systematic treatment when presenting the KKT conditions for different problem types [1, 2, 3]. For example, for optimization problems with only equality constraints, only inequality constraints, or a mixture of both, how should the corresponding geometric optimality conditions be formulated? What are the constraint qualifications in each case? What does strong duality look like in different settings? These questions are either briefly mentioned or addressed only for a single class of problems in many textbooks, without a comprehensive collection and systematic comparison across all cases. This article aims to fill this gap by systematically organizing and presenting results that are as complete as possible, covering the theorems of alternative, geometric optimality conditions, constraint qualifications, and strong duality in their specific forms for various problem types. In particular, we give a detailed treatment of Slater's condition and its multiple scenarios. We hope that this compilation will help students build a clear theoretical framework and simultaneously serve as a convenient reference for instructors preparing their lectures.

The rest of this article is organized as follows. Section 2 introduces the theorems of alternative, which lay the theoretical foundation for deriving optimality conditions for the three types of optimization problems (those with only equality constraints, only inequality constraints, and both). Section 3 discusses the geometric optimality conditions for optimization problems in each of these cases, providing a geometric interpretation of the necessary conditions that an optimal solution must satisfy. Section 4 presents the constraint qualifications that ensure the validity of the KKT conditions for each type of problem, specifying the premises under which they are effective. Section 5 systematically elaborates on strong duality for all three cases, including the conditions under which it holds and a discussion of the duality gap. Finally, Section 6 concludes the paper.

2. Theorems of Alternative

Theorems of alternative are needed to derive conditions for optimality and strong duality. According to whether there are linear inequalities or linear equations, we have the following result [4].

Suppose $g_1, \dots, g_m$ are convex functions on $X \subseteq \mathbb{R}^n$, $g_{m+1}, \dots, g_{m+k}$ are affine functions, and $h_1, \dots, h_l$ are also affine functions, then in each of the following four cases, one and only one statement in (i) and (ii) can hold.

- (1)
 - (i) There exists $\hat{\mathbf{x}} \in X$ such that $g_i(\hat{\mathbf{x}}) < 0$, $i = 1, \dots, m$.
 - (ii) There exists $\mathbf{0} \neq (\lambda_1, \dots, \lambda_m) \in \mathbb{R}_+^m$ such that $\sum_{i=1}^m \lambda_i g_i(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in X$.
- (2) Suppose there exists $\tilde{\mathbf{x}} \in \text{ri}(X)$ satisfying $g_i(\tilde{\mathbf{x}}) \leq 0$, $i = m+1, \dots, m+k$.
 - (i) There exists $\hat{\mathbf{x}} \in X$ such that $g_i(\hat{\mathbf{x}}) < 0$, $i = 1, \dots, m$, $g_i(\hat{\mathbf{x}}) \leq 0$, $i = m+1, \dots, m+k$.
 - (ii) There exists $\mathbf{0} \neq (\lambda_1, \dots, \lambda_m) \in \mathbb{R}_+^m$, $(\lambda_{m+1}, \dots, \lambda_{m+k}) \in \mathbb{R}_+^k$ such that $\sum_{i=1}^{m+k} \lambda_i g_i(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in X$.
- (3) Suppose there exists $\tilde{\mathbf{x}} \in \text{ri}(X)$ satisfying $h_j(\tilde{\mathbf{x}}) = 0$, $j = 1, \dots, l$.
 - (i) There exists $\hat{\mathbf{x}} \in X$ such that $g_i(\hat{\mathbf{x}}) < 0$, $i = 1, \dots, m$, $h_j(\hat{\mathbf{x}}) = 0$, $j = 1, \dots, l$.
 - (ii) There exists $\mathbf{0} \neq (\lambda_1, \dots, \lambda_m) \in \mathbb{R}_+^m$, $(\mu_1, \dots, \mu_l) \in \mathbb{R}^l$ such that $\sum_{i=1}^m \lambda_i g_i(\mathbf{x}) + \sum_{j=1}^l \mu_j h_j(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in X$.
- (4) Suppose there exists $\tilde{\mathbf{x}} \in \text{ri}(X)$ satisfying $g_i(\tilde{\mathbf{x}}) \leq 0$, $i = m+1, \dots, m+k$, $h_j(\tilde{\mathbf{x}}) = 0$, $j = 1, \dots, l$.
 - (i) There exists $\hat{\mathbf{x}} \in X$ such that $g_i(\hat{\mathbf{x}}) < 0$, $i = 1, \dots, m$, $g_i(\hat{\mathbf{x}}) \leq 0$, $i = m+1, \dots, m+k$, $h_j(\hat{\mathbf{x}}) = 0$, $j = 1, \dots, l$.
 - (ii) There exists $\mathbf{0} \neq (\lambda_1, \dots, \lambda_m) \in \mathbb{R}_+^m$, $(\lambda_{m+1}, \dots, \lambda_{m+k}) \in \mathbb{R}_+^k$, $(\mu_1, \dots, \mu_l) \in \mathbb{R}^l$ such that $\sum_{i=1}^{m+k} \lambda_i g_i(\mathbf{x}) + \sum_{j=1}^l \mu_j h_j(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in X$.

From the above we can derive a result of alternative concerning systems of linear inequalities. In each of the following four cases, one and only one system has a solution.

- (1)
$$\begin{cases} \mathbf{Ax} < \mathbf{0} \\ \mathbf{A}^T \mathbf{y} = \mathbf{0}, \mathbf{y} \geq \mathbf{0}, \mathbf{y} \neq \mathbf{0} \end{cases}$$
- (2)
$$\begin{cases} \mathbf{Ax} < \mathbf{0}, \mathbf{Bx} \leq \mathbf{0} \\ \mathbf{A}^T \mathbf{y} + \mathbf{B}^T \mathbf{z} = \mathbf{0}, \mathbf{y} \geq \mathbf{0}, \mathbf{y} \neq \mathbf{0}, \mathbf{z} \geq \mathbf{0} \end{cases}$$

$$(3) \begin{cases} \mathbf{Ax} < \mathbf{0}, \mathbf{Cx} = \mathbf{0} \\ \mathbf{A}^T \mathbf{y} + \mathbf{C}^T \mathbf{w} = \mathbf{0}, \mathbf{y} \geq \mathbf{0}, \mathbf{y} \neq \mathbf{0} \end{cases}$$

$$(4) \begin{cases} \mathbf{Ax} < \mathbf{0}, \mathbf{Bx} \leq \mathbf{0}, \mathbf{Cx} = \mathbf{0} \\ \mathbf{A}^T \mathbf{y} + \mathbf{B}^T \mathbf{z} + \mathbf{C}^T \mathbf{w} = \mathbf{0}, \mathbf{y} \geq \mathbf{0}, \mathbf{y} \neq \mathbf{0}, \mathbf{z} \geq \mathbf{0} \end{cases}$$

Note that case (1) is called Gordan's lemma.

3. Geometric Optimality Conditions for Constrained Optimization

According to whether there are inequality constraints, we can write the following two kinds of constrained optimization problems. The first one has only equality constraints:

$$\begin{aligned} & \min f(\mathbf{x}) \\ & \text{s.t. } h_j(\mathbf{x}) = 0, \quad j = 1, \dots, l \\ & \mathbf{x} \in X. \end{aligned} \quad (1)$$

The second one has both equality constraints and inequality constraints:

$$\begin{aligned} & \min f(\mathbf{x}) \\ & \text{s.t. } g_i(\mathbf{x}) \leq 0, \quad i = 1, \dots, m \\ & \quad h_j(\mathbf{x}) = 0, \quad j = 1, \dots, l \\ & \mathbf{x} \in X. \end{aligned} \quad (2)$$

It is understood that $f, g_1, \dots, g_m, h_1, \dots, h_l$ are all defined on $X \subseteq \mathbb{R}^n$. We use $I(\mathbf{x}^*)$ to denote the indices of active inequality constraints, that is $I(\mathbf{x}^*) = \{i | g_i(\mathbf{x}^*) = 0\}$.

Suppose $f, g_1, \dots, g_m$ are differentiable at $\mathbf{x}^* \in \text{int}(X)$, $h_1, \dots, h_l$ have continuous partial derivatives in a neighbourhood of $\mathbf{x}^*$, then we have the following geometric optimality conditions.

(1) Suppose $\mathbf{x}^*$ is a local minimizer of problem (1). If $\nabla h_1(\mathbf{x}^*), \dots, \nabla h_l(\mathbf{x}^*)$ are linearly independent, then there does not exist $\mathbf{d} \in \mathbb{R}^n$ satisfying $\nabla f(\mathbf{x}^*)^T \mathbf{d} < 0$, $\nabla h_j(\mathbf{x}^*)^T \mathbf{d} = 0, j = 1, \dots, l$.

(2) Suppose $\mathbf{x}^*$ is a local minimizer of problem (2). If $\nabla h_1(\mathbf{x}^*), \dots, \nabla h_l(\mathbf{x}^*)$ are linearly independent, then there does not exist $\mathbf{d} \in \mathbb{R}^n$ satisfying $\nabla f(\mathbf{x}^*)^T \mathbf{d} < 0$, $\nabla g_i(\mathbf{x}^*)^T \mathbf{d} < 0, i \in I(\mathbf{x}^*)$, $\nabla h_j(\mathbf{x}^*)^T \mathbf{d} = 0, j = 1, \dots, l$.

The implicit function theorem is needed to prove the above result. If there is no equality constraints or the equality constraint functions are affine, then similar result can be proved without the help of the implicit function theorem. Specifically, we consider the following constrained optimization problems.

The first one is a constrained optimization problems with only inequality constraints $g_i(\mathbf{x}) \leq 0, i = 1, \dots, m$:

$$\begin{aligned} & \min f(\mathbf{x}) \\ & \text{s.t. } g_i(\mathbf{x}) \leq 0, \quad i = 1, \dots, m \\ & \mathbf{x} \in X. \end{aligned} \quad (3)$$

The second one has only inequality constraints besides $g_i(\mathbf{x}) \leq 0, i = 1, \dots, m$:

$$\begin{aligned} & \min f(\mathbf{x}) \\ & \text{s.t. } g_i(\mathbf{x}) \leq 0, \quad i = 1, \dots, m \\ & \quad g_i(\mathbf{x}) \leq 0, \quad i = m + 1, \dots, m + k \\ & \mathbf{x} \in X. \end{aligned} \quad (4)$$

The third one has only equality constraints besides $g_i(\mathbf{x}) \leq 0, i = 1, \dots, m$:

$$\begin{aligned} & \min f(\mathbf{x}) \\ & \text{s.t. } g_i(\mathbf{x}) \leq 0, \quad i = 1, \dots, m \\ & \quad h_j(\mathbf{x}) = 0, \quad j = 1, \dots, l \\ & \mathbf{x} \in X. \end{aligned} \quad (5)$$

The last one has both equality constraints and inequality constraints besides $g_i(\mathbf{x}) \leq 0, i = 1, \dots, m$:

$$\begin{aligned} & \min f(\mathbf{x}) \\ & \text{s.t. } g_i(\mathbf{x}) \leq 0, \quad i = 1, \dots, m \\ & \quad g_i(\mathbf{x}) \leq 0, \quad i = m + 1, \dots, m + k \\ & \quad h_j(\mathbf{x}) = 0, \quad j = 1, \dots, l \\ & \mathbf{x} \in X. \end{aligned} \quad (6)$$

Suppose $f, g_1, \dots, g_m$ are differentiable at $\mathbf{x}^* \in \text{int}(X)$, $g_{m+1}, \dots, g_{m+k}$ are affine functions, and $h_1, \dots, h_l$ are also affine functions, then we have the following geometric optimality conditions.

(1) Suppose $\mathbf{x}^*$ is a local minimizer of problem (3), then there does not exist $\mathbf{d} \in \mathbb{R}^n$ satisfying $\nabla f(\mathbf{x}^*)^T \mathbf{d} < 0$, $\nabla g_i(\mathbf{x}^*)^T \mathbf{d} < 0, i \in I(\mathbf{x}^*)$.

(2) Suppose $\mathbf{x}^*$ is a local minimizer of problem (4), then there does not exist $\mathbf{d} \in \mathbb{R}^n$ satisfying $\nabla f(\mathbf{x}^*)^T \mathbf{d} < 0$, $\nabla g_i(\mathbf{x}^*)^T \mathbf{d} < 0, i \in I(\mathbf{x}^*) \cap \{1, \dots, m\}$, $\nabla g_i(\mathbf{x}^*)^T \mathbf{d} \leq 0, i \in I(\mathbf{x}^*) \cap \{m + 1, \dots, m + k\}$, $\nabla h_j(\mathbf{x}^*)^T \mathbf{d} = 0, j = 1, \dots, l$.

(3) Suppose $\mathbf{x}^*$ is a local minimizer of problem (5), then there does not exist $\mathbf{d} \in \mathbb{R}^n$ satisfying $\nabla f(\mathbf{x}^*)^T \mathbf{d} < 0$, $\nabla g_i(\mathbf{x}^*)^T \mathbf{d} < 0, i \in I(\mathbf{x}^*)$, $\nabla h_j(\mathbf{x}^*)^T \mathbf{d} = 0, j = 1, \dots, l$.

(4) Suppose $\mathbf{x}^*$ is a local minimizer of problem (6), then there does not exist $\mathbf{d} \in \mathbb{R}^n$ satisfying $\nabla f(\mathbf{x}^*)^T \mathbf{d} < 0$, $\nabla g_i(\mathbf{x}^*)^T \mathbf{d} < 0, i \in I(\mathbf{x}^*) \cap \{1, \dots, m\}$, $\nabla g_i(\mathbf{x}^*)^T \mathbf{d} \leq 0, i \in I(\mathbf{x}^*) \cap \{m + 1, \dots, m + k\}$, $\nabla h_j(\mathbf{x}^*)^T \mathbf{d} = 0, j = 1, \dots, l$.

4. Constraint Qualifications for KKT Conditions

From geometric optimality conditions in Section 3, with the help of theorems of alternative in Section 2, we can derive KKT conditions. Suppose $f, g_1, \dots, g_m$ are differentiable at $\mathbf{x}^* \in \text{int}(X)$, $h_1, \dots, h_l$ have continuous partial derivatives in a neighbourhood of $\mathbf{x}^*$, then we have the following optimality conditions.

(1) Suppose $\mathbf{x}^*$ is a local minimizer of problem (3), if $\nabla g_i(\mathbf{x}^*), i \in I(\mathbf{x}^*)$ are linearly independent, then there exists a $\lambda \in \mathbb{R}_+^m$ such that $\mathbf{x}^*, \lambda$ satisfy KKT conditions.

(2) Suppose $\mathbf{x}^*$ is a local minimizer of problem (1), if $\nabla h_j(\mathbf{x}^*), j = 1, \dots, l$ are linearly independent, then there exists a $\mu \in \mathbb{R}^l$ such that $\mathbf{x}^*, \mu$ satisfy KKT conditions.

(3) Suppose $\mathbf{x}^*$ is a local minimizer of problem (2), if $\nabla g_i(\mathbf{x}^*), i \in I(\mathbf{x}^*)$, $\nabla h_j(\mathbf{x}^*), j = 1, \dots, l$ are linearly independent, then there exist $\lambda \in \mathbb{R}_+^m, \mu \in \mathbb{R}^l$ such that $\mathbf{x}^*, \lambda, \mu$ satisfy KKT conditions.

For constrained optimization problems with convex inequality constraints and linear equality constraints, Slater's condition is easier to use compared with the above constraint qualification. Suppose $f, g_1, \dots, g_m$ are differentiable at $\mathbf{x}^* \in \text{int}(X)$, $g_1, \dots, g_m$ are convex functions on X , $g_{m+1}, \dots, g_{m+k}$ are affine functions, and $h_1, \dots, h_l$ are also affine functions, then we have the following optimality conditions.

(1) Suppose $\mathbf{x}^*$ is a local minimizer of problem (3), if there

exists $\hat{\mathbf{x}} \in X$ satisfying $g_i(\hat{\mathbf{x}}) < 0$, $i = 1, \dots, m$, then there exists a $\lambda \in \mathbb{R}_+^m$ such that $\mathbf{x}^*, \lambda$ satisfy KKT conditions.

(2) Suppose $\mathbf{x}^*$ is a local minimizer of problem (4), if there exists $\hat{\mathbf{x}} \in X$ satisfying $g_i(\hat{\mathbf{x}}) < 0$, $i = 1, \dots, m$, $g_i(\hat{\mathbf{x}}) \leq 0$, $i = m+1, \dots, m+k$, then there exists a $\lambda \in \mathbb{R}_+^{m+k}$ such that $\mathbf{x}^*, \lambda$ satisfy KKT conditions.

(3) Suppose $\mathbf{x}^*$ is a local minimizer of problem (5), if there exists $\hat{\mathbf{x}} \in X$ satisfying $g_i(\hat{\mathbf{x}}) < 0$, $i = 1, \dots, m$, $h_j(\hat{\mathbf{x}}) = 0$, $j = 1, \dots, l$, then there exist $\lambda \in \mathbb{R}_+^m, \mu \in \mathbb{R}^l$ such that $\mathbf{x}^*, \lambda, \mu$ satisfy KKT conditions.

(4) Suppose $\mathbf{x}^*$ is a local minimizer of problem (6), if there exists $\hat{\mathbf{x}} \in X$ satisfying $g_i(\hat{\mathbf{x}}) < 0$, $i = 1, \dots, m$, $g_i(\hat{\mathbf{x}}) \leq 0$, $i = m+1, \dots, m+k$, $h_j(\hat{\mathbf{x}}) = 0$, $j = 1, \dots, l$, then there exist $\lambda \in \mathbb{R}_+^{m+k}, \mu \in \mathbb{R}^l$ such that $\mathbf{x}^*, \lambda, \mu$ satisfy KKT conditions.

For constrained optimization problems with linear constraints, Slater's condition is not required. Suppose f is differentiable at $\mathbf{x}^* \in \text{int}(X)$, $g_1, \dots, g_m$ are affine functions, and $h_1, \dots, h_l$ are also affine functions, then we have the following optimality conditions.

(1) Suppose $\mathbf{x}^*$ is a local minimizer of problem (3), then there exists a $\lambda \in \mathbb{R}_+^m$ such that $\mathbf{x}^*, \lambda$ satisfy KKT conditions.

(2) Suppose $\mathbf{x}^*$ is a local minimizer of problem (1), then there exists a $\mu \in \mathbb{R}^l$ such that $\mathbf{x}^*, \mu$ satisfy KKT conditions.

(3) Suppose $\mathbf{x}^*$ is a local minimizer of problem (2), then there exist $\lambda \in \mathbb{R}_+^m, \mu \in \mathbb{R}^l$ such that $\mathbf{x}^*, \lambda, \mu$ satisfy KKT conditions.

5. Strong Duality

The Lagrangian of problem (2) is defined as

$$L(\mathbf{x}, \lambda, \mu) = f(\mathbf{x}) + \sum_{i=1}^m \lambda_i g_i(\mathbf{x}) + \sum_{j=1}^l \mu_j h_j(\mathbf{x}), \mathbf{x} \in X, \lambda \in \mathbb{R}_+^m, \mu \in \mathbb{R}^l,$$

where $\lambda = (\lambda_1, \dots, \lambda_m)$, $\mu = (\mu_1, \dots, \mu_l)$. Let

$$\begin{aligned} \tilde{f}(\mathbf{x}) &= \sup_{(\lambda, \mu) \in \mathbb{R}_+^m \times \mathbb{R}^l} L(\mathbf{x}, \lambda, \mu), \\ q(\lambda, \mu) &= \inf_{\mathbf{x} \in X} L(\mathbf{x}, \lambda, \mu), \end{aligned}$$

then problem (2) can be rewritten as

$$\min_{\mathbf{x}} \tilde{f}(\mathbf{x})$$

$$\text{s.t. } \mathbf{x} \in X,$$

and the dual problem is defined as

$$\max_{\lambda, \mu} q(\lambda, \mu)$$

$$\text{s.t. } (\lambda, \mu) \in \mathbb{R}_+^m \times \mathbb{R}^l.$$

Dual problems for optimization problems (1), (3), (4), (5), (6) can be defined analogously. Note that for problem (1) which has only equality constraints, the Lagrangian should be written as $L(\mathbf{x}, \mu)$ and the dual objective function should be written as $q(\mu)$. Similarly, for problem (3) which has only inequality constraints, the Lagrangian should be written as $L(\mathbf{x}, \lambda)$ and the dual objective function should be written as $q(\lambda)$.

The infimum of the primal problem is always not smaller than the supremum of the dual problem, which is called weak duality. If these two values coincide, then strong duality holds. For constrained optimization problems with convex inequality constraints and linear equality constraints, strong duality holds when Slater's condition is satisfied. Suppose $f, g_1, \dots, g_m$ are convex on X , $g_{m+1}, \dots, g_{m+k}$ are affine

functions, and $h_1, \dots, h_l$ are also affine functions, then we have the following results concerning strong duality.

(1) If there exists $\hat{\mathbf{x}} \in X$ for which $g_i(\hat{\mathbf{x}}) < 0$, $i = 1, \dots, m$, then strong duality holds for optimization problem (3) and the dual problem attains its maximum.

(2) If there exists $\hat{\mathbf{x}} \in \text{ri}(X)$ for which $g_i(\hat{\mathbf{x}}) < 0$, $i = 1, \dots, m$, $g_i(\hat{\mathbf{x}}) \leq 0$, $i = m+1, \dots, m+k$, then strong duality holds for optimization problem (4) and the dual problem attains its maximum.

(3) If there exists $\hat{\mathbf{x}} \in \text{ri}(X)$ for which $g_i(\hat{\mathbf{x}}) < 0$, $i = 1, \dots, m$, $h_j(\hat{\mathbf{x}}) = 0$, $j = 1, \dots, l$, then strong duality holds for optimization problem (5) and the dual problem attains its maximum.

(4) If there exists $\hat{\mathbf{x}} \in \text{ri}(X)$ for which $g_i(\hat{\mathbf{x}}) < 0$, $i = 1, \dots, m$, $g_i(\hat{\mathbf{x}}) \leq 0$, $i = m+1, \dots, m+k$, $h_j(\hat{\mathbf{x}}) = 0$, $j = 1, \dots, l$, then strong duality holds for optimization problem (6) and the dual problem attains its maximum.

For constrained optimization problems with linear constraints, strong duality holds under weaker conditions than the above. Suppose f is convex on X , $g_1, \dots, g_m$ are affine functions, and $h_1, \dots, h_l$ are also affine functions, then we have the following results concerning strong duality.

(1) If there exists $\hat{\mathbf{x}} \in \text{ri}(X)$ for which $g_i(\hat{\mathbf{x}}) \leq 0$, $i = 1, \dots, m$, then strong duality holds for optimization problem (3) and the dual problem attains its maximum.

(2) If there exists $\hat{\mathbf{x}} \in \text{ri}(X)$ for which $h_j(\hat{\mathbf{x}}) = 0$, $j = 1, \dots, l$, then strong duality holds for optimization problem (1) and the dual problem attains its maximum.

(3) If there exists $\hat{\mathbf{x}} \in \text{ri}(X)$ for which $g_i(\hat{\mathbf{x}}) \leq 0$, $i = 1, \dots, m$, $h_j(\hat{\mathbf{x}}) = 0$, $j = 1, \dots, l$, then strong duality holds for optimization problem (2) and the dual problem attains its maximum.

6. KKT Conditions as Sufficient and Necessary Conditions for Convex Optimization Problems

For convex optimization problems, KKT conditions are not only necessary but also sufficient for global minimization under Slater's condition. Suppose $f, g_1, \dots, g_m$ are differentiable at $\mathbf{x}^* \in \text{int}(X)$, $f, g_1, \dots, g_m$ are convex functions on X , $g_{m+1}, \dots, g_{m+k}$ are affine functions, and $h_1, \dots, h_l$ are also affine functions, then we have the following optimality conditions.

(1) If there exists $\hat{\mathbf{x}} \in X$ for which $g_i(\hat{\mathbf{x}}) < 0$, $i = 1, \dots, m$, then $\mathbf{x}^*$ is a global minimizer of problem (3), if and only if there exists a $\lambda \in \mathbb{R}_+^m$ such that $\mathbf{x}^*, \lambda$ satisfy KKT conditions.

(2) If there exists $\hat{\mathbf{x}} \in X$ for which $g_i(\hat{\mathbf{x}}) < 0$, $i = 1, \dots, m$, $g_i(\hat{\mathbf{x}}) \leq 0$, $i = m+1, \dots, m+k$, then $\mathbf{x}^*$ is a global minimizer of problem (4), if and only if there exists a $\lambda \in \mathbb{R}_+^{m+k}$ such that $\mathbf{x}^*, \lambda$ satisfy KKT conditions.

(3) If there exists $\hat{\mathbf{x}} \in X$ for which $g_i(\hat{\mathbf{x}}) < 0$, $i = 1, \dots, m$, $h_j(\hat{\mathbf{x}}) = 0$, $j = 1, \dots, l$, then $\mathbf{x}^*$ is a global minimizer of problem (5), if and only if there exist $\lambda \in \mathbb{R}_+^m, \mu \in \mathbb{R}^l$ such that $\mathbf{x}^*, \lambda, \mu$ satisfy KKT conditions.

(4) If there exists $\hat{\mathbf{x}} \in X$ for which $g_i(\hat{\mathbf{x}}) < 0$, $i = 1, \dots, m$, $g_i(\hat{\mathbf{x}}) \leq 0$, $i = m+1, \dots, m+k$, $h_j(\hat{\mathbf{x}}) = 0$, $j = 1, \dots, l$, then $\mathbf{x}^*$ is a global minimizer of problem (6), if and only if there exist $\lambda \in \mathbb{R}_+^{m+k}, \mu \in \mathbb{R}^l$ such that $\mathbf{x}^*, \lambda, \mu$ satisfy KKT conditions.

For convex optimization problems with linear constraints, we have simpler results. Suppose f is differentiable at $\mathbf{x}^* \in$

$\text{int}(X)$, f is convex on X , $g_1, \dots, g_m$ are affine functions, and $h_1, \dots, h_l$ are also affine functions, then we have the following optimality conditions.

(1) $\mathbf{x}^*$ is a global minimizer of problem (3), if and only if there exists a $\boldsymbol{\lambda} \in \mathbb{R}_+^m$ such that $\mathbf{x}^*, \boldsymbol{\lambda}$ satisfy KKT conditions.

(2) $\mathbf{x}^*$ is a global minimizer of problem (1), if and only if there exists a $\boldsymbol{\mu} \in \mathbb{R}^l$ such that $\mathbf{x}^*, \boldsymbol{\mu}$ satisfy KKT conditions.

(3) $\mathbf{x}^*$ is a global minimizer of problem (2), if and only if there exist $\boldsymbol{\lambda} \in \mathbb{R}_+^m, \boldsymbol{\mu} \in \mathbb{R}^l$ such that $\mathbf{x}^*, \boldsymbol{\lambda}, \boldsymbol{\mu}$ satisfy KKT conditions.

7. Conclusion

We have presented a systematic compilation of optimality conditions for constrained optimization problems, covering the theorems of alternative, geometric optimality conditions, constraint qualifications, KKT conditions, and strong duality in a unified manner across three problem types: those with only equality constraints, only inequality constraints, and a mixture of both. By organizing the material around these categories and paying special attention to the various cases involving Slater's condition, we aim to provide students with a clear and comprehensive reference that bridges the gaps

often left by existing textbooks. This exposition not only clarifies the logical relationships among these fundamental results but also equips students with the necessary theoretical tools to tackle a wide range of optimization problems. Ultimately, we hope that this work will enhance students' understanding of constrained optimization theory, strengthen their analytical skills, and serve as a valuable resource for instructors in their teaching.

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References

- [1] Beck, A. (2023). *Introduction to nonlinear optimization: Theory, algorithms, and applications with python and matlab* (2nd ed.). SIAM.
- [2] Nocedal, J., & Wright, S. J. (2006). *Numerical optimization* (2nd ed.). Springer.
- [3] Avriel, M. (2003). *Nonlinear programming: Analysis and methods*. Dover.
- [4] Rockafellar, R. T. (1970). *Convex analysis*. Princeton University Press.